

# THE BREZIS-NIRENBERG PROBLEM FOR FRACTIONAL ELLIPTIC OPERATORS

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ABSTRACT. Let  $\mathcal{L} = \operatorname{div}(A(x)\nabla)$  be a uniformly elliptic operator in divergence form in a bounded open subset  $\Omega$  of  $\mathbb{R}^n$ . We study the effect of the operator  $\mathcal{L}$  on the existence and nonexistence of positive solutions of the nonlocal Brezis-Nirenberg problem

$$\begin{cases} (-\mathcal{L})^s u &= u^{\frac{n+2s}{n-2s}} + \lambda u & \text{in } \Omega, \\ u &= 0 & \text{on } \mathbb{R}^n \setminus \Omega \end{cases}$$

where  $(-\mathcal{L})^s$  denotes the fractional power of  $-\mathcal{L}$  with zero Dirichlet boundary values on  $\partial\Omega$ ,  $0 < s < 1$ ,  $n > 2s$  and  $\lambda$  is a real parameter. By assuming  $A(x) \geq A(x_0)$  for all  $x \in \overline{\Omega}$  and  $A(x) \leq A(x_0) + |x - x_0|^\sigma I_n$  near some point  $x_0 \in \overline{\Omega}$ , we prove existence theorems for any  $\lambda \in (0, \lambda_{1,s}(-\mathcal{L}))$ , where  $\lambda_{1,s}(-\mathcal{L})$  denotes the first Dirichlet eigenvalue of  $(-\mathcal{L})^s$ . Our existence result holds true for  $\sigma > 2s$  and  $n \geq 4s$  in the interior case ( $x_0 \in \Omega$ ) and for  $\sigma > \frac{2s(n-2s)}{n-4s}$  and  $n > 4s$  in the boundary case ( $x_0 \in \partial\Omega$ ). Nonexistence for star-shaped domains is obtained for any  $\lambda \leq 0$ .

## 1. INTRODUCTION AND STATEMENTS

A lot of attention has been paid to a number of counterparts of the Brezis-Nirenberg problem since the pioneer paper [5] which consists in determining all values of  $\lambda$  for which the problem

$$(1.1) \quad \begin{cases} -\Delta u &= u^{\frac{n+2}{n-2}} + \lambda u & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega \end{cases}$$

admits a positive solution, where  $\Omega$  is a bounded open subset of  $\mathbb{R}^n$ ,  $n \geq 3$  and  $\lambda$  is a real parameter.

According to [5], the problem (1.1) admits a positive solution for any  $\lambda \in (0, \lambda_1(-\Delta))$  provided that  $n \geq 4$ , where  $\lambda_1(-\Delta)$  denotes the first Dirichlet eigenvalue of  $-\Delta$  on  $\Omega$ . Moreover, the problem has no such solution for  $n \geq 3$  if either  $\lambda \geq \lambda_1(-\Delta)$  or  $\lambda \leq 0$  and  $\Omega$  is a star-shaped  $C^1$  domain. When  $n = 3$  and  $\Omega$  is a ball, a positive solution of (1.1) exists if, and only if,  $\lambda \in (\frac{1}{4}\lambda_1(-\Delta), \lambda_1(-\Delta))$ .

The Brezis-Nirenberg problem for uniformly elliptic operators in divergence form has been studied in the works [14, 16, 18]. Precisely, consider the problem

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$$(1.2) \quad \begin{cases} -\mathcal{L}u &= u^{\frac{n+2}{n-2}} + \lambda u & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega \end{cases}$$

where  $\mathcal{L} = \operatorname{div}(A(x)\nabla)$ . Assume that  $A(x) = (a_{ij}(x))$  is a positive definite symmetric matrix for each  $x \in \overline{\Omega}$  with continuous entries on  $\overline{\Omega}$ , so that  $\mathcal{L}$  is a selfadjoint uniformly elliptic operator. Assume also there exist a point  $x_0 \in \overline{\Omega}$  and a constant  $C_0 > 0$  such that

$$(1.3) \quad A(x) \geq A(x_0) \quad \text{for every } x \in \overline{\Omega}$$

and

$$(1.4) \quad A(x) \leq A(x_0) + C_0|x - x_0|^\sigma I_n \quad \text{locally around } x_0$$

both in the sense of bilinear forms, where  $I_n$  denotes the  $n \times n$  identity matrix. In [16], Egnell focused on the interior case ( $x_0 \in \Omega$ ) and proved that problem (1.2) admits a positive solution for any  $\lambda \in (0, \lambda_1(-\mathcal{L}))$  provided that  $n \geq 4$  and  $\sigma > 2$ , where  $\lambda_1(-\mathcal{L})$  denotes the first Dirichlet eigenvalue of  $-\mathcal{L}$  on  $\Omega$ . The boundary case ( $x_0 \in \partial\Omega$ ) has recently been treated in [18], which proves the existence of a positive solution for any  $\lambda \in (0, \lambda_1(-\mathcal{L}))$  provided that  $n > 4$ ,  $\sigma > \frac{2n-4}{n-4}$  and the boundary of  $\Omega$  is  $\alpha$ -singular at  $x_0$ , with  $\alpha \in [1, \sigma \frac{n-4}{2n-4})$ , in the following sense:

**Definition 1.** *The boundary of an open subset  $\Omega$  of  $\mathbb{R}^n$  is said to be  $\alpha$ -singular at the point  $x_0$ , with  $\alpha \geq 1$ , if there exist a constant  $\delta > 0$  and a sequence  $(x_j) \subset \Omega$  such that  $x_j \rightarrow x_0$  as  $j \rightarrow +\infty$  and  $B(x_j, \delta|x_j - x_0|^\alpha) \subseteq \Omega$ .*

The nonexistence of positive solutions of (1.2) for  $\lambda \leq 0$  on star-shaped domains has been proved in [16] (see also [14]) by assuming  $a_{ij} \in C^1(\overline{\Omega} \setminus \{x_0\})$  such that  $a'_{ij}(x) := \nabla a_{ij}(x) \cdot (x - x_0)$  extends continuously to  $x_0$  and  $A'(x) = (a'_{ij}(x))$  is positive semi-definite for every  $x \in \Omega$ . Nonexistence of positive solution in the case  $\lambda \geq \lambda_1(-\mathcal{L})$  follows from a standard argument.

When  $0 < s < 1$ , Caffarelli and Silvestre [8] introduced the characterization of the fractional power of the Laplace operator  $(-\Delta)^s$  in terms of a Dirichlet-to-Neumann map associated to a suitable extension problem. Since then, a great deal of attention has been dedicated in the last years to nonlinear nonlocal problems involving this operator. See for example [2, 3, 7, 10, 12, 13, 25, 26], among others. Two of them (see [25] for  $s = \frac{1}{2}$  and [2] for other values of  $s \in (0, 1)$ ) consider the following counterpart of the Brezis-Nirenberg problem

$$(1.5) \quad \begin{cases} (-\Delta)^s u &= u^{\frac{n+2s}{n-2s}} + \lambda u & \text{in } \Omega, \\ u &= 0 & \text{in } \mathbb{R}^n \setminus \Omega \end{cases}$$

In particular, it has been proved that problem (1.5) admits a positive viscosity solution for any  $\lambda \in (0, \lambda_{1,s}(-\Delta))$  provided that  $n \geq 4s$ , where  $\lambda_{1,s}(-\Delta)$  denotes the first Dirichlet eigenvalue of  $(-\Delta)^s$  on  $\Omega$ . Moreover, there exists no such solution in  $C^1(\overline{\Omega})$  for  $n > 2s$  if either  $\lambda \leq 0$  and  $\Omega$  is a star-shaped  $C^1$  domain or  $\lambda \geq \lambda_{1,s}(-\Delta)$ .

This work dedicates special attention to the effect of the elliptic operator  $\mathcal{L}$  (or of the matrix  $A(x)$ ) on the existence and nonexistence of positive viscosity solutions of the following Brezis-Nirenberg problem involving the fractional power of  $-\mathcal{L}$ ,

$$(1.6) \quad \begin{cases} (-\mathcal{L})^s u &= u^{\frac{n+2s}{n-2s}} + \lambda u & \text{in } \Omega, \\ u &= 0 & \text{in } \mathbb{R}^n \setminus \Omega \end{cases}$$

Denote by  $\lambda_{1,s}(-\mathcal{L})$  the first Dirichlet eigenvalue of  $(-\mathcal{L})^s$  on  $\Omega$ .

The main existence theorems are:

**Theorem 1.** (*Interior case*) Let  $0 < s < 1$  and  $\Omega \subset \mathbb{R}^n$  be a bounded open set. Assume entries of the matrix  $A$  are continuous in  $\overline{\Omega}$  and  $A$  satisfies (1.3), (1.4) for some  $x_0 \in \Omega$ . Then (1.6) admits at least one positive weak solution for any  $\lambda \in (0, \lambda_{1,s}(-\mathcal{L}))$  provided  $n \geq 4s$  and  $\sigma > 2s$ . If  $\partial\Omega$  is of  $C^{1,1}$  class and each entry of  $A(x)$  belongs to  $C^1(\overline{\Omega})$ , then our weak solution  $u$  belongs to  $C^{0,\alpha}(\overline{\Omega})$  if  $0 < s < 1/2$  and to  $C^{1,\alpha}(\overline{\Omega})$  if  $1/2 \leq s < 1$  for any  $0 < \alpha < 1$ .

**Theorem 2.** (*Boundary case*) Let  $0 < s < 1$  and  $\Omega \subset \mathbb{R}^n$  be a bounded open set. Assume entries of the matrix  $A$  are continuous in  $\overline{\Omega}$  and  $A$  satisfies (1.3), (1.4) for some  $x_0$  on  $\partial\Omega$ . Suppose  $\partial\Omega$  is  $\alpha$ -singular at  $x_0$ . Then (1.6) admits at least one positive weak solution for any  $\lambda \in (0, \lambda_{1,s}(-\mathcal{L}))$  if  $n > 4s$ ,  $\sigma > \frac{2s(n-2s)}{n-4s}$  and  $1 \leq \alpha < \frac{\sigma(n-4s)}{2s(n-2s)}$ . If  $\partial\Omega$  is of  $C^{1,1}$  class and each entry of  $A(x)$  belongs to  $C^1(\overline{\Omega})$ , then our weak solution  $u$  belongs to  $C^{0,\alpha}(\overline{\Omega})$  if  $0 < s < 1/2$  and to  $C^{1,\alpha}(\overline{\Omega})$  if  $1/2 \leq s < 1$  for any  $0 < \alpha < 1$ .

The nonexistence theorem states that

**Theorem 3.** Let  $0 < s < 1$ ,  $n > 2s$ , and  $\Omega \subset \mathbb{R}^n$  be a bounded open set which is star-shaped with respect to some point  $x_0 \in \overline{\Omega}$  and its boundary is of  $C^1$  class. Assume matrix  $A$  satisfies (1.3)-(1.4) for some  $x_0$  in  $\overline{\Omega}$ , moreover, assume  $a_{ij} \in C^1(\overline{\Omega} \setminus \{x_0\})$  and  $a'_{ij}(x) := \nabla a_{ij}(x) \cdot (x - x_0)$  extends continuously to  $x_0$  and  $A'(x) = (a'_{ij}(x))$  is positive semi-definite for every  $x \in \Omega$ . Then (1.6) admits no positive solution in  $C^1(\overline{\Omega})$  for any  $\lambda \leq 0$ . Furthermore, if  $\partial\Omega$  is of  $C^{1,1}$  class and each entry of  $A(x)$  belongs to  $C^1(\overline{\Omega})$ , then (1.6) admits no positive weak solution for any  $\lambda \leq 0$  provided that  $s \geq 1/2$ .

Theorems 1 and 3 extend the existence and nonexistence results of [2] and [25], since the constant matrix  $A(x) = I_n$  clearly fulfills our assumptions. All results in [16, 18] for  $s = 1$  are also extended fully to any  $0 < s < 1$ . One prototype example of operator  $\mathcal{L}$  is  $A(x) = A_0 + |x - x_0|^\sigma I_n$ .

A natural approach for solving (1.6) consists in searching for minimizers of the functional

$$u \mapsto \int_{\mathbb{R}^n} |(-\mathcal{L})^{s/2} u|^2 - \lambda \int_{\Omega} u^2 dx$$

subject to

$$\int_{\Omega} |u|^{\frac{2n}{n-2s}} dx = 1.$$

However, fractional integrals of this type are generally difficult to be handled directly. On the other hand, fractional powers of elliptic operators in divergence form were recently described in [23] as Dirichlet-to-Neumann maps for an extension problem in the spirit of the extension problem for the fractional Laplace operator on  $\mathbb{R}^n$  of [8]. In section 2, we take some advantage of this description and provide

an equivalent variational formulation which will be used in our proof of existence. We also present an existence tool and a regularity result of weak solutions.

The existence of minimizers for the new constrained functional often relies on the construction and estimates of suitable bubbles involving extremal functions of Sobolev type inequalities. Although, this is a well known strategy, new and important difficulties arise in present context. Indeed, the most delicate part in the proof of Theorem 1 (the interior case) is caused by the term  $|x - x_0|^\sigma$  in the inequality (1.4), which essentially involves estimates of the multiple integral

$$\int_{B_R(0)} \int_0^\infty |x|^\sigma y^{1-2s} |\nabla_x w_1(x, y)|^2 dy dx$$

on the whole ball of radius  $R$  for  $R > 0$  large enough, where  $w_1(x, y)$  is given by

$$w_1(x, y) := c_s y^{2s} \int_{\mathbb{R}^n} \frac{u_1(\xi)}{(|x - \xi|^2 + y^2)^{\frac{n+2s}{2}}} d\xi,$$

with  $c_s$  being an appropriate normalization constant and

$$u_1(\xi) = \frac{1}{(1 + |\xi|^2)^{\frac{n-2s}{2}}}.$$

In Section 3, we estimate the integral mentioned above and, as a byproduct, we prove Theorem 1. The bubbles used in the proof of Theorem 1 do not work in the boundary case because we need to compare the least energy level to the corresponding best trace constant in  $\mathbb{R}_+^{n+1}$ . The idea for overcoming this difficulty is to consider suitable bubbles concentrated in interior points converging fast to the boundary point  $x_0$  in an appropriate way. The construction depends on the order  $\alpha$  of the singularity of the boundary at  $x_0$ . In Section 4 we introduce such bubbles and derive the necessary estimates in the boundary case. Proof of Theorem 2 then follows. Finally, in Section 5, we establish a Pohozaev identity for  $C^1$  solutions of (1.6) and use it to prove Theorem 3.

## 2. THE VARIATIONAL FRAMEWORK AND MAIN TOOLS

For the precise definition of the fractional power of the selfadjoint elliptic operator  $-\mathcal{L}$ , we consider an orthonormal basis of  $L^2(\Omega)$  consisting of eigenfunctions  $\phi_k \in H_0^1(\Omega)$ ,  $k = 1, 2, \dots$ , that correspond to eigenvalues  $\lambda_1 < \lambda_2 \leq \dots$ . The domain of  $(-\mathcal{L})^s$ , denoted here by  $H^s$ ,  $0 < s < 1$ , is defined as the Hilbert space of functions  $u = \sum_{k=1}^\infty c_k \phi_k \in L^2(\Omega)$  such that  $\sum_{k=1}^\infty \lambda_k^s c_k^2 < \infty$  endowed with the inner product  $\langle u, v \rangle_{H^s} := \sum_{k=1}^\infty \lambda_k^s c_k d_k$ , where  $v = \sum_{k=1}^\infty d_k \phi_k$ . For each  $u = \sum_{k=1}^\infty c_k \phi_k \in H^s$ , we define  $(-\mathcal{L})^s u = \sum_k c_k \lambda_k^s \phi_k$ . With this definition, we have

$$\langle u, v \rangle_{H^s} = \langle (-\mathcal{L})^{\frac{s}{2}} u, (-\mathcal{L})^{\frac{s}{2}} v \rangle_{L^2(\Omega)}.$$

It is well known that (see details in [9, 21])

$$H^s = \begin{cases} H^s(\Omega), & \text{if } 0 < s < 1/2, \\ H_{00}^{1/2}(\Omega), & \text{if } s = 1/2, \\ H_0^s(\Omega), & \text{if } 1/2 < s < 1 \end{cases}$$

The spaces  $H^s(\Omega)$  and  $H_0^s(\Omega)$ ,  $s \neq 1/2$ , are the classical fractional Sobolev spaces given as the completion of  $C_0^\infty(\Omega)$  under the norm

$$\|u\|_{H^s(\Omega)}^2 = \|u\|_{L^2(\Omega)}^2 + [u]_{H^s(\Omega)}^2,$$

where

$$[u]_{H^s(\Omega)}^2 = \int_{\Omega} \int_{\Omega} \frac{(u(x) - u(y))^2}{|x - y|^{n+2s}} dx dy.$$

The space  $H_{00}^{1/2}(\Omega)$  is the Lions-Magenes space which consists of functions  $u \in L^2(\Omega)$  such that  $[u]_{H^{1/2}(\Omega)}^2 < \infty$  and

$$\int_{\Omega} \frac{u(x)^2}{\text{dist}(x, \partial\Omega)} dx < \infty.$$

The Hilbert space  $H^s$  is compactly embedded in  $L^q(\Omega)$  for any  $1 \leq q < \frac{2n}{n-2s}$  and continuously in  $L^{\frac{2n}{n-2s}}(\Omega)$  provided that  $n > 2s$ . So, a natural strategy to solve (1.6) in a weak sense consists in searching minimizers of

$$(2.1) \quad I_{\lambda}^A(u) = \int_{\Omega} \left| (-\mathcal{L})^{\frac{s}{2}} u \right|^2 dx - \lambda \int_{\Omega} u^2 dx$$

constrained to the Nehari manifold

$$E = \left\{ u \in H^s : \int_{\Omega} |u|^{\frac{2n}{n-2s}} dx = 1 \right\}.$$

In fact, the functional  $I_{\lambda}^A$  is well defined on  $H^s$  and its least energy level on  $E$ , denoted by

$$S_{\lambda}^A := \inf_{u \in E} I_{\lambda}^A(u),$$

is finite. Moreover, a minimizer  $u \in H^s$  satisfies

$$(2.2) \quad \int_{\Omega} (-\mathcal{L})^{\frac{s}{2}} u (-\mathcal{L})^{\frac{s}{2}} \zeta dx = \int_{\Omega} \left( S_{\lambda}^A |u|^{\frac{4s}{n-2s}} u + \lambda u \right) \zeta dx$$

for every  $\zeta \in H^s$ . In particular, if  $S_{\lambda}^A$  is positive and  $u$  is nonnegative in  $\Omega$ , then  $u$  is a nonnegative weak solution of (1.6). On the other hand, using the variational characterization of the first Dirichlet eigenvalue of  $(-\mathcal{L})^s$  given by

$$\lambda_{1,s}(-\mathcal{L}) = \inf_{u \in H \setminus \{0\}} \frac{\int_{\Omega} \left| (-\mathcal{L})^{\frac{s}{2}} u \right|^2 dx}{\int_{\Omega} u^2 dx},$$

one can easily check that the positivity of  $S_{\lambda}^A$  is equivalent to  $\lambda < \lambda_{1,s}(-\mathcal{L})$ . So, the conclusion of Theorems 1 and 2 follows if we are able to prove the existence and regularity of nonnegative minimizers of  $I_{\lambda}^A$  in  $E$  for any  $\lambda > 0$ .

Inspired by the recent work in [23], an equivalent definition for the operator  $(-\mathcal{L})^s$  in  $\Omega$  with zero Dirichlet boundary condition can be formulated as an extension problem in a cylinder. Let  $\mathcal{C}_{\Omega} = \Omega \times (0, \infty) \subset \mathbb{R}_+^{n+1}$ . We denote the points in  $\mathcal{C}_{\Omega}$  by  $z = (x, y)$  with  $x \in \Omega$  and the lateral boundary  $\partial\Omega \times [0, \infty)$  by  $\partial_L \mathcal{C}_{\Omega}$ . Then for  $u \in H^s$ , we define the  $(s, A)$ -extension  $w = E_s^A(u)$  as the solution to the problem

$$(2.3) \quad \begin{cases} \operatorname{div} (y^{1-2s} B(x) \nabla w(x, y)) &= 0 & \text{in } \mathcal{C}_\Omega, \\ w &= 0 & \text{on } \partial_L \mathcal{C}_\Omega, \\ w &= u & \text{on } \Omega \times \{y = 0\}. \end{cases}$$

Here  $B(x) = \begin{pmatrix} A(x) & 0 \\ 0 & 1 \end{pmatrix}$  is an  $(n+1) \times (n+1)$  matrix. The extension function belongs to the space

$$H_{0,L}^{s,A}(\mathcal{C}_\Omega) = \overline{C_0^\infty(\Omega \times [0, \infty))}^{\|\cdot\|_{H_{0,L}^{s,A}}}$$

with

$$\|w\|_{H_{0,L}^{s,A}} = \left( c_s \iint_{\mathcal{C}_\Omega} y^{1-2s} (\nabla w)^T B(x) \nabla w \, dx dy \right)^{1/2}.$$

Here  $c_s$  is a normalization constant such that  $E_s^A : H^s(\Omega) \rightarrow H_{0,L}^{s,A}(\mathcal{C}_\Omega)$  is an isometry between Hilbert spaces. In particular,

$$\|E_s^A u\|_{H_{0,L}^{s,A}} = \|u\|_{H^s}$$

for every  $u \in H^s$ .

For  $A(x) = I_n$ ,  $E_s^A(u)$  is the canonical  $s$ -harmonic extension of  $u$ , see [8]. In this case, we denote  $E_s^A(u)$  by  $E_s(u)$  and the extension function space  $H_{0,L}^{s,A}(\mathcal{C}_\Omega)$  is denoted by  $H_{0,L}^s(\mathcal{C}_\Omega)$ .

It is known from [23] that the extension function  $w$  satisfies

$$(2.4) \quad -c_s \lim_{y \rightarrow 0^+} y^{1-2s} \frac{\partial w}{\partial y}(x, y) = (-\mathcal{L})^s u(x)$$

for every  $x \in \Omega$ .

Using the extension map  $E_s^A$ , we can reformulate the problem (1.6) as

$$(2.5) \quad \begin{cases} -\operatorname{div} (y^{1-2s} B(x) \nabla w) &= 0 & \text{in } \mathcal{C}_\Omega, \\ w &= 0 & \text{on } \partial \mathcal{C}_\Omega, \\ \partial_\nu^s w &= w^{\frac{n+2s}{n-2s}} + \lambda w & \text{in } \Omega \times \{y = 0\}. \end{cases}$$

Here

$$\partial_\nu^s w := -c_s \lim_{y \rightarrow 0^+} y^{1-2s} \frac{\partial w}{\partial y}.$$

Using that the trace of functions in  $H_{0,L}^{s,A}(\mathcal{C}_\Omega)$  is compactly embedded in  $L^q(\Omega)$  for any  $1 \leq q < \frac{2n}{n-2s}$  and continuously in  $L^{\frac{2n}{n-2s}}(\Omega)$  for  $n > 2s$ , we can consider the minimization of functional

$$(2.6) \quad J_\lambda^A(w) = c_s \iint_{\mathcal{C}_\Omega} y^{1-2s} (\nabla w)^T B(x) \nabla w \, dx dy - \lambda \int_\Omega w^2(x, 0) \, dx$$

in the admissible set

$$F = \left\{ w \in H_{0,L}^{s,A}(\mathcal{C}_\Omega) : \int_\Omega |w(x, 0)|^{\frac{2n}{n-2s}} \, dx = 1 \right\}.$$

Clearly, we have

$$S_\lambda^A = \inf_{w \in F} J_\lambda^A(w).$$

Moreover,  $u \in E$  is a minimizer of  $I_\lambda^A$  on  $E$  if, and only if,  $w = E_s^A(u) \in F$  is a minimizer of  $J_\lambda^A$  on  $F$ .

There are essentially two advantages in considering the minimization problem for  $J_\lambda^A$ . Firstly, it follows directly that  $|w| \in F$  and  $J_\lambda^A(|w|) = J_\lambda^A(w)$  for every  $w \in F$ , so that minimizers of  $J_\lambda^A$  on  $F$  can be assumed nonnegative in  $\Omega$ . Secondly, the integral

$$\iint_{\mathcal{C}_\Omega} y^{1-2s} (\nabla w)^T B(x) \nabla w \, dx dy$$

is more easily to be handled comparing to the one in (2.1). Therefore, from now on we will concentrate on the existence of minimizers of  $J_\lambda^A$  in  $F$ .

One of the tool used in our existence proof is the following trace inequality

$$(2.7) \quad \left( \int_{\Omega} |f(x, 0)|^r \, dx \right)^{\frac{2}{r}} \leq A \int_{\mathcal{C}_\Omega} y^{1-2s} |\nabla f(x, y)|^2 \, dx dy$$

for  $1 \leq r \leq \frac{2n}{n-2s}$ ,  $n > 2s$  and every  $f \in H_0^s(\mathcal{C}_\Omega)$ . When  $r = \frac{2n}{n-2s}$ , we denote the best constant in (2.7) by  $K_s(n)$ . This constant is not achieved in bounded domain and achieved when  $\Omega = \mathbb{R}^n$  and  $f = E_s(u)$  with

$$(2.8) \quad u(x) = \frac{\varepsilon^{\frac{n-2s}{2}}}{(|x|^2 + \varepsilon^2)^{\frac{n-2s}{2}}}.$$

By a change of variable argument, we have

$$(2.9) \quad \left( \int_{\Omega} |f(x, 0)|^{\frac{2n}{n-2s}} \, dx \right)^{\frac{n-2s}{n}} \leq \det(A(x_0))^{-\frac{s}{n}} K_s(n) \int_{\mathcal{C}_\Omega} y^{1-2s} (\nabla f(x, y))^T B(x_0) \nabla f(x, y) \, dx dy$$

for every  $f \in H_0^s(\mathcal{C}_\Omega)$ .

The following proposition states a necessary condition for existence of minimizer of  $I_\lambda^A$  in  $E$ .

**Proposition 1.** *Let  $n > 2s$ . Assume there exists a point  $x_0 \in \overline{\Omega}$  such that (1.3) is satisfied and*

$$(2.10) \quad S_\lambda^A < c_s \det(A(x_0))^{\frac{s}{n}} K_s(n)^{-1}.$$

*Then, the infimum  $S_\lambda^A$  of  $I_\lambda^A$  in  $E$  is achieved by some nonnegative function  $u$ . Furthermore, if  $\lambda < \lambda_{1,s}(-\mathcal{L})$ , then  $u$  is a nonnegative weak solution of (1.6), module a suitable scaling.*

*Proof.* As noted above, it suffices to prove that the infimum of  $J_\lambda^A$  in  $F$ , given also by  $S_\lambda^A$ , is assumed by some nonnegative function  $w$ .

Let  $\{w_m\} \subset H_{0,L}^{s,A}(\mathcal{C}_\Omega)$  be a minimizing sequence of  $J_\lambda^A$  on  $F$ . Clearly,  $\{w_m\}$  is bounded in  $H_{0,L}^{s,A}(\mathcal{C}_\Omega)$ . Since  $\Omega$  is bounded, up to a subsequence, we have

$$\begin{aligned}
w_m &\rightharpoonup w \text{ weakly in } H_{0,L}^{s,A}(\mathcal{C}_\Omega), \\
w_m(\cdot, 0) &\rightarrow w(\cdot, 0) \text{ strongly in } L^q(\Omega) \text{ for } 1 \leq q < \frac{2n}{n-2s}, \\
w_m(\cdot, 0) &\rightarrow w(\cdot, 0) \text{ a.e. in } \Omega.
\end{aligned}$$

A direct calculation, taking into account of the weak convergence, gives

$$\begin{aligned}
\|w_m\|_{H_{0,L}^{s,A}}^2 &= \|w_m - w\|_{H_{0,L}^{s,A}}^2 + \|w\|_{H_{0,L}^{s,A}}^2 \\
&\quad + c_s \iint_{\mathcal{C}_\Omega} y^{1-2s} (\nabla w^T B(x) \nabla (w_m - w) + \nabla^T (w_m - w) B(x) \nabla w) \, dx dy \\
&= \|w_m - w\|_{H_{0,L}^{s,A}}^2 + \|w\|_{H_{0,L}^{s,A}}^2 + o(1).
\end{aligned}$$

Now using Brezis-Lieb Lemma (see [4]), (1.3) and (2.7), we obtain

$$\begin{aligned}
S_\lambda^A &= \|w_m\|_{H_{0,L}^{s,A}}^2 - \lambda \|w_m(\cdot, 0)\|_{L^2}^2 + o(1) \\
&= \|w_m - w\|_{H_{0,L}^{s,A}}^2 + \|w\|_{H_{0,L}^{s,A}}^2 - \lambda \|w(\cdot, 0)\|_{L^2}^2 + o(1) \\
&\geq c_s \iint_{\mathcal{C}_\Omega} y^{1-2s} (\nabla (w_m - w))^T B(x_0) \nabla (w_m - w) \, dx dy + J_\lambda^A(w) + o(1) \\
&\geq c_s \det(A(x_0))^{\frac{s}{n}} K_s(n)^{-1} \|(w_m - w)(\cdot, 0)\|_{L^{\frac{2n}{n-2s}}}^2 + J_\lambda^A(w) + o(1) \\
&\geq c_s \det(A(x_0))^{\frac{s}{n}} K_s(n)^{-1} \left(1 - \int_\Omega |w(x, 0)|^{\frac{2n}{n-2s}} dx\right)^{\frac{n-2s}{n}} \\
&\quad + S_\lambda^A \left(\int_\Omega |w(x, 0)|^{\frac{2n}{n-2s}} dx\right)^{\frac{n-2s}{n}} + o(1)
\end{aligned}$$

so that

$$S_\lambda^A \left(1 - \left(\int_\Omega |w(x, 0)|^{\frac{2n}{n-2s}} dx\right)^{\frac{n-2s}{n}}\right) \geq c_s \det(A(x_0))^{\frac{s}{n}} K_s(n)^{-1} \left(1 - \int_\Omega |w(x, 0)|^{\frac{2n}{n-2s}} dx\right)^{\frac{n-2s}{n}}.$$

So, using the assumption (2.10) and the fact that

$$\int_\Omega |w(x, 0)|^{\frac{2n}{n-2s}} dx \leq 1,$$

we derive

$$\int_\Omega |w(x, 0)|^{\frac{2n}{n-2s}} dx = 1.$$

But this implies that

$$w_m(\cdot, 0) \rightarrow w(\cdot, 0) \text{ strongly in } L^{\frac{2n}{n-2s}}(\Omega).$$

Then,  $w \in F$  and by lower semicontinuity of  $J_\lambda^A$ ,

$$J_\lambda^A(w) \leq \liminf_{m \rightarrow \infty} J_\lambda^A(w_m),$$

so that  $J_\lambda^A(w) = S_\lambda^A$ . Therefore,  $|w|$  is a nonnegative minimizer of  $J_\lambda^A$  in  $F$ .



The remainder of the proof is direct because  $u = |w|(\cdot, 0)$  is a nonnegative minimizer of  $I_\lambda^A$  on  $E$  and the inequality  $\lambda < \lambda_{1,s}(-\mathcal{L})$  is equivalent to the positivity of  $S_\lambda^A$ .  $\square$

The discussion made so far about the existence of nonnegative weak solutions of (1.6) can be resumed in the following remark:

**Remark 1.** *Proposition 1 provides the existence of a nonnegative weak solution of (1.6) by assuming the conditions (1.3), (2.10) and  $\lambda < \lambda_{1,s}(-\mathcal{L})$ . Up to a scaling this solution can be seen as the trace of a nonnegative minimizer of  $Q_\lambda^A$  on  $H_{0,L}^{s,A}(\mathcal{C}_\Omega) \setminus \{0\}$ , where  $Q_\lambda^A$  denotes the Rayleigh quotient*

$$Q_\lambda^A(w) = \frac{\|w\|_{H_{0,L}^{s,A}}^2 - \lambda \|w(\cdot, 0)\|_{L^2}^2}{\|w(\cdot, 0)\|_{L^{\frac{2n}{n-2s}}}^2}.$$

Note that (2.10) is equivalent to existence of a function  $w_0 \in H_{0,L}^{s,A}(\mathcal{C}_\Omega) \setminus \{0\}$  so that

$$(2.11) \quad Q_\lambda^A(w_0) < c_s \det(A(x_0))^{\frac{s}{n}} K_s(n)^{-1}.$$

So, in light of Proposition 1, Sections 3 and 4 are dedicated to the construction of  $w_0$  by using the remaining assumptions assumed in Theorems 1 and 2, respectively.

The next proposition shows further regularity of weak solution of (1.6).

**Proposition 2.** *Let  $u \in H^s \setminus \{0\}$  be a nonnegative weak solution of (1.6). Assume that  $A(x)$  is a positive definite symmetric matrix for each  $x \in \overline{\Omega}$  with continuous entries on  $\overline{\Omega}$ . Then  $u \in L^p(\Omega)$  for every  $p \geq 1$ . Furthermore, if  $\partial\Omega$  is of  $C^{1,1}$  class and each entry of  $A(x) \in C^1(\overline{\Omega})$  then  $u$  belongs to  $C^{0,\alpha}(\overline{\Omega})$  if  $0 < s < 1/2$  and to  $C^{1,\alpha}(\overline{\Omega})$  if  $1/2 \leq s < 1$  for any  $0 < \alpha < 1$ .*

*Proof.* The function  $w = E_s^A(u)$  satisfies (2.5). Since  $u$  is assumed to be nonnegative,  $w$  is also nonnegative. For each  $k \geq 1$ , we define  $w_k$  by

$$w_k(x, y) := \min\{w(x, y), k\}.$$

Since  $ww_k^{2\beta}$  is in  $H_{0,L}^{s,A}(\mathcal{C}_\Omega)$  for all  $\beta \geq 0$ , using it as a test function in (2.5), we obtain

$$\begin{aligned} (2.12) \quad & \int_{\Omega} f(u) w w_k^{2\beta} dx \\ &= \iint_{\mathcal{C}_\Omega} y^{1-2s} (\nabla w)^T B(x) \nabla (w w_k^{2\beta}) dx dy \\ &= \iint_{\mathcal{C}_\Omega} y^{1-2s} w_k^{2\beta} (\nabla w)^T B(x) \nabla w + 2\beta y^{1-2s} w_k^{2\beta} (\nabla w_k)^T B(x) \nabla w_k dx dy, \end{aligned}$$

where  $f(u) := u^{\frac{n+2s}{n-2s}} + \lambda u$ . Note that the last equality comes from the fact the  $w = w_k$  in the set where  $w \leq k$  and  $w_k$  is a constant otherwise.

On the other hand, we have

$$\begin{aligned}
(2.13) \quad & \iint_{\mathcal{C}_\Omega} y^{1-2s} (\nabla(w w_k^\beta))^T B(x) \nabla(w w_k^\beta) \, dx dy \\
&= \iint_{\mathcal{C}_\Omega} y^{1-2s} w_k^{2\beta} (\nabla w)^T B(x) \nabla w + (2\beta + \beta^2) y^{1-2s} w_k^{2\beta} (\nabla w_k)^T B(x) \nabla w_k \, dx dy.
\end{aligned}$$

Combining (2.12) and (2.13), we derive

$$(2.14) \quad \iint_{\mathcal{C}_\Omega} y^{1-2s} |\nabla(w w_k^\beta)|^2 \, dx dy \leq C_1 \int_\Omega |f(u)| u w_k^{2\beta} \, dx$$

for some constant  $C_1 > 0$  which depends only on  $\beta$  and the matrix  $A$ . Let  $h = |f(u)|/(1+u)$  and  $\Omega_m = \{x \in \Omega : u(x) > m\}$ .

Since  $h \in L^{\frac{n}{2s}}(\Omega)$ , there exists  $m \in \mathbb{N}$  large enough such that

$$\left( \int_{\Omega_m} |h|^{\frac{n}{2s}} \, dx \right)^{\frac{2s}{n}} \leq \frac{K_s(n)}{4C_1},$$

where  $K_s(n)$  is the best constant with respect to the embedding  $H_{0,L}^s(\mathcal{C}_\Omega) \hookrightarrow L^{\frac{2n}{n-2s}}(\Omega)$ . Since  $u$  is bounded on  $\Omega \setminus \Omega_m$ , there exists a constant  $C_2 > 0$  which depends only on  $m$  and  $\Omega$  such that

$$\begin{aligned}
& \int_\Omega |f(u)| u w_k^{2\beta} \, dx = \int_{\Omega \setminus \Omega_m} |f(u)| u w_k^{2\beta} \, dx + \int_{\Omega_m} |f(u)| u w_k^{2\beta} \, dx \\
(2.15) \quad & \leq C_2 + 2 \int_{\Omega_m} h u^2 w_k^{2\beta} \, dx
\end{aligned}$$

Using (2.15), we deduce that

$$\begin{aligned}
& \int_\Omega |f(u)| u w_k^{2\beta} \, dx \leq C_2 + 2 \left( \int_{\Omega_m} h^{\frac{n}{2s}} \, dx \right)^{\frac{2s}{n}} \left( \int_{\Omega_m} (u w_k^\beta)^{\frac{2n}{n-2s}} \, dx \right)^{\frac{n-2s}{n}} \\
& \leq C_2 + \frac{1}{2C_1} \iint_{\mathcal{C}_\Omega} y^{1-2s} |\nabla(w w_k^\beta)|^2 \, dx dy.
\end{aligned}$$

Plugging this in (2.14) and taking  $k \rightarrow \infty$  we have  $u^{\beta+1} \in L^{\frac{2n}{n-2s}}(\Omega)$  for all  $\beta \geq 0$ . Thus  $f(u) \in L^p(\Omega)$  for every  $p \geq 1$  and the rest of the proof follows from Theorems 1.1, 1.2, 1.3 and 1.5 of [9].  $\square$

### 3. PROOF OF THEOREM 1 - INTERIOR CASE

Throughout this section, we assume  $x_0$  is an interior point of  $\Omega$ . According to Remark 1 and Proposition 2 of the previous section, our main task in this section is to prove the following proposition:

**Proposition 3.** *Assume (1.4) for some  $x_0 \in \Omega$ . If  $n \geq 4s$  and  $\sigma > 2s$ , then for any  $\lambda > 0$  there exists  $w_0 \in H_{0,L}^{s,A}(\mathcal{C}_\Omega) \setminus \{0\}$  such that*

$$Q_\lambda^A(w_0) < c_s \det(A(x_0))^{\frac{s}{n}} K_s(n)^{-1}.$$

We first derive some necessary estimate. For simplicity of notations, we first assume  $x_0 = 0$  and  $A(0) = I_n$ . Choose a smooth nonincreasing cut-off function  $\phi(t) \in C^\infty(\mathbb{R}_+)$  such that

$$\phi(t) = 1 \text{ for } 0 \leq t \leq \frac{1}{2} \text{ and } \phi(t) = 0 \text{ if } t \geq 1.$$

Let  $r$  be small enough so that  $B(0, r) \subset \Omega$ . Define  $\phi_r(x, y) = \phi\left(\frac{rxy}{r}\right)$  with  $r_{xy} = \left(|x|^2 + y^2\right)^{\frac{1}{2}}$ . Let  $u_\varepsilon$  be given by (2.8) and  $w_\varepsilon = E_s(u_\varepsilon)$ . Then  $w_\varepsilon(x, y) = \varepsilon^n w_1\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right)$ .

**Lemma 1.** *With the above notations, the family  $\{\phi_r w_\varepsilon\}_{\varepsilon>0}$  and its trace on  $\{y=0\}$ , namely  $\{\phi_r u_\varepsilon\}_{\varepsilon>0}$  satisfy*

$$(3.1) \quad \|\phi_r w_\varepsilon\|_{H_{0,L}^{s,A}(C_\Omega)}^2 \leq c_s \int_{\mathbb{R}_+^{n+1}} y^{1-2s} |\nabla w_\varepsilon(x, y)|^2 dx dy \\ + \begin{cases} O(\varepsilon^\sigma) + O(\varepsilon^{n-2s}), & \text{if } \sigma < n-2s \\ O(\varepsilon^\sigma \ln \frac{1}{\varepsilon}) + O(\varepsilon^{n-2s}), & \text{if } \sigma = n-2s \\ O(\varepsilon^{n-2s}), & \text{if } \sigma > n-2s \end{cases}$$

$$(3.2) \quad \|\phi_r w_\varepsilon(x, 0)\|_{L^2(\Omega)}^2 = \begin{cases} C\varepsilon^{2s} + O(\varepsilon^{n-2s}) & \text{if } n > 4s, \\ C\varepsilon^{2s} \ln \frac{1}{\varepsilon} + O(\varepsilon^{2s}) & \text{if } n = 4s. \end{cases}$$

*Proof.* The equation (3.2) follows from Lemma 3.8 in [2]. We only need to prove (3.1). Since

$$(3.3) \quad \begin{aligned} \|\phi_r w_\varepsilon\|_{H_{0,L}^{s,A}(C_\Omega)}^2 &= c_s \iint_{C_\Omega} y^{1-2s} [\nabla(\phi_r w_\varepsilon)]^T B(x) \nabla(\phi_r w_\varepsilon) dx dy \\ &= c_s \iint_{C_\Omega} y^{1-2s} \phi_r^2 (\nabla w_\varepsilon)^T B(x) \nabla w_\varepsilon dx dy \\ &\quad + 2c_s \iint_{C_\Omega} y^{1-2s} \phi_r w_\varepsilon (\nabla \phi_r)^T B(x) \nabla w_\varepsilon dx dy \\ &\quad + c_s \iint_{C_\Omega} y^{1-2s} w_\varepsilon^2 (\nabla \phi_r)^T B(x) \nabla \phi_r dx dy \\ &\leq c_s \iint_{C_\Omega} y^{1-2s} \phi_r^2 (\nabla w_\varepsilon)^T B(0) \nabla w_\varepsilon dx dy \\ &\quad + c_s \iint_{C_\Omega} y^{1-2s} \phi_r^2 |\nabla w_\varepsilon|^2 |x|^\sigma dx dy \\ &\quad + Cc_s \iint_{C_\Omega} y^{1-2s} |\phi_r| |\nabla \phi_r| |w_\varepsilon| |\nabla w_\varepsilon| dx dy \\ &\quad + Cc_s \iint_{C_\Omega} y^{1-2s} |\nabla \phi_r|^2 |w_\varepsilon|^2 dx dy \\ &= I + II + III + IV. \end{aligned}$$

The third and fourth term in (3.3) can be estimated in the same way as in proof of Lemma 3.8 in [2] and we have

$$III + IV \leq O(\varepsilon^{n-2s}).$$

The first term in (3.3) can be estimated as

$$(3.4) \quad c_s \iint_{C_\Omega} y^{1-2s} \phi_r^2 (\nabla w_\varepsilon)^T B(0) \nabla w_\varepsilon dx dy \leq c_s \int_{\mathbb{R}_+^{n+1}} y^{1-2s} |\nabla w_\varepsilon(x, y)|^2 dx dy.$$

For the rest of the proof, we estimate the second term in (3.3). We shall show

$$(3.5) \quad \iint_{\mathcal{C}_\Omega} y^{1-2s} \phi_r^2 |\nabla w_\varepsilon|^2 |x|^\sigma dx dy \leq \begin{cases} O(\varepsilon^\sigma) & \text{if } \sigma < n - 2s, \\ O\left(\varepsilon^\sigma \ln \frac{1}{\varepsilon}\right) & \text{if } \sigma = n - 2s, \\ O(\varepsilon^{n-2s}) & \text{if } \sigma > n - 2s. \end{cases}$$

To prove (3.5), we write

$$(3.6) \quad \begin{aligned} & \iint_{\mathcal{C}_\Omega} y^{1-2s} \phi_r^2 |\nabla w_\varepsilon|^2 |x|^\sigma dx dy \\ & \leq \varepsilon^{2s-n-2} \iint_{\{r_{xy} \leq r\}} y^{1-2s} \left| \nabla w_1 \left( \frac{x}{\varepsilon}, \frac{y}{\varepsilon} \right) \right|^2 |x|^\sigma dx dy \\ & = \varepsilon^\sigma \iint_{\{r_{xy} \leq \frac{r}{\varepsilon}\}} y^{1-2s} |x|^\sigma |\nabla w_1(x, y)|^2 dx dy \\ & \leq C\varepsilon^\sigma \iint_{\{r_{xy} \leq \frac{r}{\varepsilon}\} \cap \{|x| < 1\}} y^{1-2s} |\nabla w_1(x, y)|^2 dx dy \\ & \quad + C\varepsilon^\sigma \iint_{\{r_{xy} \leq \frac{r}{\varepsilon}\} \cap \{|x| \geq 1\}} y^{1-2s} |x|^\sigma |\nabla w_1(x, y)|^2 dx dy. \end{aligned}$$

We shall prove

$$(3.7) \quad \begin{aligned} & \iint_{\{|x| \geq 1\} \cap \{r_{xy} \leq \frac{r}{\varepsilon}\}} y^{1-2s} |x|^\sigma |\nabla w_1(x, y)|^2 dx dy \\ & \leq \begin{cases} C & \text{if } \sigma < n - 2s, \\ C \ln \frac{1}{\varepsilon} & \text{if } \sigma = n - 2s, \\ C\varepsilon^{n-2s-\sigma} & \text{if } \sigma > n - 2s. \end{cases} \end{aligned}$$

Then (3.5) follows directly from (3.6) and (3.7). We estimate  $\nabla_x w_1(x, y)$  using its explicit representation formula. Since

$$\begin{aligned} \nabla_{x_i} w_1(x, y) &= \int_{\mathbb{R}^n} \frac{y^{2s} (x_i - \xi_i)}{(y^2 + |x - \xi|^2)^{\frac{n+2s}{2}+1} (1 + |\xi|^2)^{\frac{n-2s}{2}}} d\xi \\ &= \int_{|\xi| < \frac{|x|}{2}} \frac{y^{2s} (x_i - \xi_i)}{(y^2 + |x - \xi|^2)^{\frac{n+2s}{2}+1} (1 + |\xi|^2)^{\frac{n-2s}{2}}} d\xi \\ & \quad + \int_{|\xi| > \frac{3|x|}{2}} \frac{y^{2s} (x_i - \xi_i)}{(y^2 + |x - \xi|^2)^{\frac{n+2s}{2}+1} (1 + |\xi|^2)^{\frac{n-2s}{2}}} d\xi \\ & \quad + \int_{\frac{|x|}{2} < |\xi| < \frac{3|x|}{2}} \frac{y^{2s} (x_i - \xi_i)}{(y^2 + |x - \xi|^2)^{\frac{n+2s}{2}+1} (1 + |\xi|^2)^{\frac{n-2s}{2}}} d\xi \\ &= I_1 + I_2 + I_3 \end{aligned}$$

If  $|\xi| < \frac{|x|}{2}$  or  $|\xi| > \frac{3|x|}{2}$ , we have  $|x - \xi| > \frac{|x|}{2}$  and  $|x - \xi| > \min(1, \frac{1}{3})|\xi|$ . We can bound  $I_1$  and  $I_2$  as follows.

$$\begin{aligned}
|I_1| &= \left| \int_{|\xi| < \frac{|x|}{2}} \frac{y^{2s}(x_i - \xi_i)}{(y^2 + |x - \xi|^2)^{\frac{n+2s}{2}+1} (1 + |\xi|^2)^{\frac{n-2s}{2}}} d\xi \right| \\
&\leq C \frac{y^{-1+2s}}{(y^2 + |x|^2)^{\frac{n+2s}{2}}} \int_{|\xi| < \frac{|x|}{2}} \frac{1}{(1 + |\xi|^2)^{\frac{n-2s}{2}}} d\xi \\
(3.8) \quad &\leq C \frac{y^{-1+2s}}{(y^2 + |x|^2)^{\frac{n+2s}{2}}} |x|^{2s},
\end{aligned}$$

$$\begin{aligned}
|I_2| &= \left| \int_{|\xi| > \frac{3|x|}{2}} \frac{y^{2s}(x_i - \xi_i)}{(y^2 + |x - \xi|^2)^{\frac{n+2s}{2}+1} (1 + |\xi|^2)^{\frac{n-2s}{2}}} d\xi \right| \\
&\leq C \frac{y^{-1+2s}}{(y^2 + |x|^2)^{\frac{n-\delta}{2}}} \int_{|\xi| > \frac{3|x|}{2}} \frac{1}{(1 + |\xi|^2)^{\frac{n-2s}{2}} |\xi|^{2s+\delta}} d\xi \\
(3.9) \quad &\leq C \frac{y^{-1+2s}}{(y^2 + |x|^2)^{\frac{n-\delta}{2}}} |x|^{-\delta}
\end{aligned}$$

Lastly, we bound  $I_3$  in two different cases. If  $y \geq |x|$ , we bound  $I_3$  by

$$\begin{aligned}
|I_3| &= \left| \int_{\frac{|x|}{2} < |\xi| < \frac{3|x|}{2}} \frac{y^{2s}(x_i - \xi_i)}{(y^2 + |x - \xi|^2)^{\frac{n+2s}{2}+1} (1 + |\xi|^2)^{\frac{n-2s}{2}}} d\xi \right| \\
&\leq \frac{C y^{2s - \frac{n+2s}{2} - 1}}{(1 + |x|^2)^{\frac{n-2s}{2}}} \int_{\frac{|x|}{2} < |\xi| < \frac{3|x|}{2}} |x - \xi|^{-\frac{n+2s}{2}} d\xi \\
&\leq \frac{C y^{-\frac{n-2s}{2} - 1}}{(1 + |x|^2)^{\frac{n-2s}{2}}} \int_{|\xi - x| < \frac{5|x|}{2}} |x - \xi|^{-\frac{n+2s}{2}} d\xi \\
&\leq C \frac{y^{-\frac{n-2s}{2} - 1}}{(1 + |x|^2)^{\frac{n-2s}{2}}} |x|^{\frac{n-2s}{2}} \\
(3.10) \quad &\leq C y^{-\frac{n-2s}{2} - 1} |x|^{-\frac{n-2s}{2}},
\end{aligned}$$

if  $y \leq |x|$ , we write

$$\begin{aligned}
I_3 &= \int_{\frac{|x|}{2} < |\xi| < \frac{3|x|}{2}} D_{x_i} P_y^s (x - \xi) f(\xi) d\xi \\
&= - \int_{\frac{|x|}{2} < |\xi| < \frac{3|x|}{2}} D_{\xi_i} P_y^s (x - \xi) f(\xi) d\xi \\
&= \int_{\frac{|x|}{2} < |\xi| < \frac{3|x|}{2}} P_y^s (x - \xi) D_{\xi_i} f(\xi) d\xi \\
&\quad - \int_{\{|\xi| = \frac{|x|}{2}\} \cup \{|\xi| = \frac{3|x|}{2}\}} P_y^s (x - \xi) f(\xi) dS_\xi \\
&= J_1 + J_2
\end{aligned}$$

$$\begin{aligned}
|J_1| &= \left| \int_{\frac{|x|}{2} < |\xi| < \frac{3|x|}{2}} P_y^s (x - \xi) D_{\xi_i} f(\xi) d\xi \right| \\
&\leq \frac{C}{\left(1 + |x|^2\right)^{\frac{n-2s+1}{2}}} \int_{\frac{|x|}{2} < |\xi| < \frac{3|x|}{2}} P_y^s (x - \xi) d\xi \\
(3.11) \quad &\leq \frac{C}{\left(1 + |x|^2\right)^{\frac{n-2s+1}{2}}}
\end{aligned}$$

When  $|\xi| = \frac{|x|}{2}$  or  $\frac{3|x|}{2}$ ,

$$(3.12) \quad P_y^s (x - \xi) f(\xi) \leq C \frac{y^{2s}}{\left(y^2 + |x|^2\right)^{\frac{n+2s}{2}}} \frac{1}{\left(1 + |x|^2\right)^{\frac{n-2s}{2}}}$$

When  $\sigma < n - 2s$ , we have the following bound from (3.8)

$$\begin{aligned}
&\int_{|x| \geq 1} \int_{|x|}^{\infty} y^{1-2s} |x|^\sigma I_1^2 dx dy \\
&\leq C \int_{|x| \geq 1} \int_{|x|}^{\infty} y^{1-2s} |x|^\sigma \frac{y^{2(2s-1)}}{\left(y^2 + |x|^2\right)^{n+2s}} |x|^{4s} dy dx \\
&\leq C \int_{|x| \geq 1} |x|^{\sigma+4s} \int_{|x|}^{\infty} y^{1-2s} \frac{y^{2(2s-1)}}{(y|x|)^{n+2s}} dy dx \\
&\leq C \int_{|x| \geq 1} \int_{|x|}^{\infty} |x|^{\sigma-n+2s} y^{-n-1} dy dx \\
&\leq C \int_{|x| \geq 1} |x|^{\sigma-2n+2s} dx \\
(3.13) \quad &= C \int_1^\infty r^{\sigma-n+2s-1} dr < \infty,
\end{aligned}$$

and

$$\begin{aligned}
& \int_{|x| \geq 1} \int_0^{|x|} y^{1-2s} |x|^\sigma I_1^2 dx dy \\
& \leq C \int_{|x| \geq 1} |x|^{4s+\sigma} \int_0^{|x|} y^{1-2s} \frac{y^{2(2s-1)}}{(y^2 + |x|^2)^{n+2s}} dy dx \\
& \leq C \int_{|x| \geq 1} |x|^{\sigma+4s-2n-4s} \int_0^{|x|} y^{2s-1} dy dx \\
(3.14) \quad & = C \int_1^\infty r^{\sigma-n+2s-1} dr < \infty.
\end{aligned}$$

Similarly, (3.9) implies

$$\begin{aligned}
& \int_{|x| \geq 1} \int_0^\infty y^{1-2s} |x|^\sigma I_2^2 dx dy \\
& \leq C \int_{|x| \geq 1} \int_0^\infty y^{1-2s} |x|^\sigma \frac{y^{2(2s-1)}}{(y^2 + |x|^2)^{n-\delta}} |x|^{-2\delta} dy dx \\
& = C \int_{|x| \geq 1} |x|^{\sigma-2\delta} \int_0^\infty \frac{y^{2s-1}}{(y^2 + |x|^2)^{n-\delta}} dy dx \\
& \leq C \int_{|x| \geq 1} |x|^{\sigma-2\delta-2n+2\delta+2s-1+1} \int_0^\infty \frac{u^{2s-1}}{(1+u^2)^{n-\delta}} du dx \\
(3.15) \quad & \leq C \int_1^\infty r^{\sigma-n+2s-1} dr < \infty.
\end{aligned}$$

Lastly, it follows from (3.10) that

$$\begin{aligned}
& \int_{|x| \geq 1} \int_{|x|}^\infty y^{1-2s} |x|^\sigma I_3^2 dx dy \\
& \leq C \int_{|x| \geq 1} |x|^{\sigma-(n-2s)} \int_{|x|}^\infty y^{1-2s-(n-2s)-2} dy dx \\
& \leq C \int_{|x| \geq 1} |x|^{\sigma-(2n-2s)} dx \\
(3.16) \quad & = C \int_1^\infty r^{\sigma-n+2s-1} dr < \infty.
\end{aligned}$$

If  $\sigma < n - 2s$ , (3.11) and (3.12) imply

$$\begin{aligned}
 & \int_{|x| \geq 1} \int_0^{|x|} y^{1-2s} |x|^\sigma J_1^2 dx dy \\
 \leq & C \int_{|x| \geq 1} \frac{|x|^\sigma}{(1 + |x|^2)^{n-2s+1}} \int_0^{|x|} y^{1-2s} dy dx \\
 \leq & C \int_{|x| \geq 1} |x|^{\sigma-2(n-2s+1)} |x|^{2-2s} dx \\
 (3.17) \quad & = C \int_1^\infty r^{\sigma-n+2s-1} dr < \infty,
 \end{aligned}$$

and

$$\begin{aligned}
 & \int_{|x| \geq 1} \int_0^{|x|} y^{1-2s} |x|^\sigma J_2^2 dx dy \\
 \leq & C \int_{|x| \geq 1} |x|^\sigma \int_0^{|x|} y^{1-2s} \frac{y^{4s} |x|^{2(n-1)}}{(y^2 + |x|^2)^{n+2s} (1 + |x|^2)^{n-2s}} dy dx \\
 \leq & C \int_{|x| \geq 1} |x|^{\sigma+2(n-1)-4n} \int_0^{|x|} y^{1+2s} dy dx \\
 (3.18) \quad & = C \int_1^\infty r^{\sigma-n+2s-1} dr < \infty.
 \end{aligned}$$

It then follows from (3.13) – (3.18) that

$$\begin{aligned}
 & \int_{|x| \geq 1} \int_0^\infty y^{1-2s} |x|^\sigma |\nabla_x w_1(x, y)|^2 dy dx \\
 = & \int_{|x| \geq 1} \int_0^\infty y^{1-2s} |x|^\sigma (I_1 + I_2 + I_3)^2 dy dx \\
 \leq & C \int_{|x| \geq 1} \int_0^\infty y^{1-2s} |x|^\sigma (I_1^2 + I_2^2 + I_3^2) dy dx < \infty.
 \end{aligned}$$

For  $\sigma = n - 2s$ , the same integrals in (3.13) – (3.18) derive

$$\iint_{\{|x| \geq 1\} \cap \{r_{xy} \leq \frac{r}{\varepsilon}\}} y^{1-2s} |x|^\sigma |\nabla w_1(x, y)|^2 dy dx \leq C \ln \frac{1}{\varepsilon}.$$

For  $\sigma > n - 2s$ , integrals in (3.13) – (3.18) yield

$$\iint_{\{|x| \geq 1\} \cap \{r_{xy} \leq \frac{r}{\varepsilon}\}} y^{1-2s} |x|^\sigma |\nabla w_1(x, y)|^2 dy dx \leq C \left(\frac{1}{\varepsilon}\right)^{\sigma-n+2s} = C \varepsilon^{n-2s-\sigma}.$$

□

For general  $A(0)$  case, we consider the following coordinate transformation. Let  $\{a_i\}_{i=1}^n$  be eigenvalues of  $A(x_0)$  and  $O$  be the orthogonal matrix such that

$$A(0) = O^T \text{diag} (a_1, \dots, a_n) O.$$



Define the mapping  $\Phi : \Omega \rightarrow \tilde{\Omega} = \Phi(\Omega)$  by  $\tilde{x} = \Phi(x)$  such that  $\tilde{x}_i = a_i^{-\frac{1}{2}}(Ox)_i$ . Let  $\tilde{u}_\varepsilon(\tilde{x}) = \frac{\varepsilon^{\frac{n-2s}{2}}}{(|\tilde{x}|^2 + \varepsilon^2)^{\frac{n-2s}{2}}}$  and  $\tilde{w}_\varepsilon(\tilde{x}, y) = E_s(\tilde{u}_\varepsilon)$ . Then we have

$$\tilde{w}_\varepsilon(\tilde{x}, y) = \varepsilon^{\frac{2s-n}{2}} \tilde{w}_1\left(\frac{\tilde{x}}{\varepsilon}, \frac{y}{\varepsilon}\right).$$

For  $x = \Phi^{-1}(\tilde{x})$ , we define  $u_\varepsilon(x) = \tilde{u}_\varepsilon(\tilde{x})$  and  $w_\varepsilon(x, y) = \tilde{w}_\varepsilon(\tilde{x}, y)$ .

To construct test functions, denote  $B^{n+1}(x, r)$  the ball in  $\mathbb{R}^{n+1}$  centered at  $(x, 0)$  with the radius  $r$ . We fix  $r > 0$  small enough such that  $\overline{B^{n+1}(\tilde{x}_0, r)} \subset \tilde{\mathcal{C}}_{\tilde{\Omega}}$ . We define the cutoff function  $\phi_r(x, y) \in C_0^\infty(\Phi^{-1}(B_+^{n+1}(\tilde{x}_0, r)))$  by  $\phi_r(x, y) = \phi_0(r_{xy}/r)$  with  $r_{xy}^2 = |\Phi(x)|^2 + y^2$ , and let

$$(3.19) \quad V_\varepsilon(x, y) = \phi_r(x, y) w_\varepsilon(x, y).$$

Under these notations, we have

$$\begin{aligned} & c_s \iint_{\mathcal{C}_\Omega} y^{1-2s} \phi_r^2 (\nabla w_\varepsilon)^T B(0) \nabla w_\varepsilon dx dy \\ &= c_s \iint_{\mathcal{C}_\Omega} y^{1-2s} \phi_r^2 \left( \sum_{i=1}^n a_i (D_{x_i} w_\varepsilon)^2 + (D_y w_\varepsilon)^2 \right) dx dy \\ &= \det(A(0))^{\frac{1}{2}} c_s \int_{\mathbb{R}_+^{n+1}} y^{1-2s} |\nabla_{(\tilde{x}, y)} \tilde{w}_\varepsilon(\tilde{x}, y)|^2 d\tilde{x} dy, \\ & \|u_\varepsilon\|_{L^2(\Omega)}^2 = \det(A(0)) \|\tilde{u}_\varepsilon\|_{L^2(\tilde{\Omega})}^2, \end{aligned}$$

and

$$\|u_\varepsilon\|_{L^r(\Omega)}^r = \det(A(0))^{\frac{r}{2}} \|\tilde{u}_\varepsilon\|_{L^r(\tilde{\Omega})}^r.$$

We then have the following bounds on  $V_\varepsilon$ .

**Lemma 2.** *With the above notations, for small  $\varepsilon > 0$ , the family of functions  $\{V_\varepsilon\}_{\varepsilon>0}$  and its trace on  $\{y=0\}$ , namely  $\{V_\varepsilon(x, 0)\}_{\varepsilon>0}$  satisfy*

$$(3.20) \quad \|V_\varepsilon\|_{H_{0,L}^{s,A}(\mathcal{C}_\Omega)}^2 \leq c_s \int_{\mathbb{R}_+^{n+1}} y^{1-2s} |\nabla \tilde{w}_\varepsilon(x, y)|^2 d\tilde{x} dy + \begin{cases} O(\varepsilon^\sigma) + O(\varepsilon^{n-2s}), & \text{if } \sigma < n-2s \\ O(\varepsilon^\sigma \ln \frac{1}{\varepsilon}) + O(\varepsilon^{n-2s}), & \text{if } \sigma = n-2s \\ O(\varepsilon^{n-2s}), & \text{if } \sigma > n-2s, \end{cases}$$

$$(3.21) \quad \|V_\varepsilon(x, 0)\|_{L^2(\Omega)}^2 = \begin{cases} C\varepsilon^{2s} + O(\varepsilon^{n-2s}) & \text{if } n > 4s, \\ C\varepsilon^{2s} \ln \frac{1}{\varepsilon} + O(\varepsilon^{2s}) & \text{if } n = 4s, \end{cases}.$$

*Proof of Proposition 3.* Fix  $r$  small such that  $V_\varepsilon = \phi_r w_\varepsilon$  defined by (3.19) is in  $H_{0,L}^{s,A}(\Omega)$ . Recall that  $\tilde{x}_i = a_i^{-\frac{1}{2}}(Ox)_i$  and  $\tilde{u}_\varepsilon(\tilde{x}) = u_\varepsilon(x)$ . We have

$$\|u_\varepsilon\|_{L^{\frac{2n}{n-2s}}}^2 = \det(A(0))^{\frac{n-2s}{n}} \|\tilde{u}_\varepsilon\|_{L^{\frac{2n}{n-2s}}}^2.$$

Let  $K_1 = \|\tilde{u}_\varepsilon\|_{L^{\frac{2n}{n-2s}}}^{\frac{2n}{n-2s}}$ . Then  $K_1$  is independent of  $\varepsilon$  and by calculations in the proof of Proposition 4.1 [2],

$$(3.22) \quad \int_{\Omega} |V_\varepsilon|^{\frac{2n}{n-2s}} dx \geq K_1 \det(A(0))^{\frac{1}{2}} + O(\varepsilon^n).$$

Since  $\widetilde{w}_\varepsilon$  is an extremal function of (2.9), we have

$$K_1^{-\frac{(n-2s)}{n}} \iint_{\mathbb{R}_+^{n+1}} y^{1-2s} |\nabla \widetilde{w}_\varepsilon|^2 dx dy = K_s(n)^{-1}.$$

When  $n > 4s$ , if  $2s < \sigma < n - 2s$ ,

$$\begin{aligned} Q_\lambda^A(V_\varepsilon) &\leq \frac{c_s(\det(A(0)))^{\frac{1}{2}} \int_{\mathbb{R}_+^{n+1}} y^{1-2s} |\nabla \widetilde{w}_\varepsilon|^2 dx dy - \lambda C \varepsilon^{2s} + O(\varepsilon^{n-2s}) + O(\varepsilon^\sigma)}{K_1^{-\frac{(n-2s)}{n}} \det(A(0))^{\frac{n-2s}{2n}} + O(\varepsilon^n)} \\ &\leq \frac{c_s(\det(A(0)))^{\frac{s}{n}} K_s(n)^{-1} - \lambda C \varepsilon^{2s} K_1^{-\frac{2(n-2s)}{(n+2s)}} + O(\varepsilon^{n-2s}) + O(\varepsilon^\sigma)}{1 + O(\varepsilon^n)} \\ (3.23) &< c_s(\det(A(0)))^{\frac{s}{n}} K_s(n)^{-1} \end{aligned}$$

for  $\varepsilon \ll 1$ . If  $\sigma \geq n - 2s$ , we replace  $\varepsilon^\sigma$  by  $\varepsilon^\sigma \ln \frac{1}{\varepsilon}$  or  $\varepsilon^{n-2s}$  in (3.23), then the same estimate follows. When  $n = 2s$ , we replace  $\varepsilon^{2s}$  by  $\varepsilon^{2s} \ln \frac{1}{\varepsilon}$  in (3.23), conclusion follows from the fact that  $\sigma > 2s$ .  $\square$

#### 4. PROOF OF THEOREM 2 - BOUNDARY CASE

Throughout this section, we assume  $x_0 = 0$  is on the boundary of  $\Omega$  and  $\partial\Omega$  is  $\alpha$ -singular at  $x_0$ . Our main task is to prove (2.11) when  $n > 4s$ ,  $\sigma > \frac{2s(n-2s)}{n-4s}$ , and  $1 \leq \alpha < \frac{\sigma(n-4s)}{2s(n-2s)}$ .

We consider the mapping  $\Phi : \Omega \rightarrow \widetilde{\Omega}$  defined in Section 3 and denote  $\widetilde{x} = \Phi(x)$ . Then by Definition 1, there exist a constant  $\delta > 0$  and a sequence  $(x_j) \subset \Omega$  (i.e.  $(\widetilde{x}_j) \subset \widetilde{\Omega}$ ) such that  $x_j \rightarrow 0$  (i.e.  $\widetilde{x}_j \rightarrow 0$ ) as  $j \rightarrow +\infty$  and  $\Phi^{-1}(B(\widetilde{x}_j, \delta|\widetilde{x}_j|^\alpha)) \subset \Omega$ . Let

$$V_\varepsilon(x, y) = \phi_\delta(x, y) w_\varepsilon(x, y)$$

defined as in (3.19). For fixed  $\beta > \alpha$ , we consider

$$\widetilde{V}_j(x, y) = \phi_\delta\left(\frac{x - x_j}{\varepsilon_j^\alpha}, \frac{y}{\varepsilon_j^\alpha}\right) w_{\varepsilon_j^\beta}(x - x_j, y),$$

where  $\varepsilon_j = |x_j - x_0|^\alpha$ . Then  $\widetilde{V}_j(x, y)$  can be rewritten as

$$\begin{aligned} \widetilde{V}_j(x, y) &= (\varepsilon_j^\alpha)^{\frac{2s-n}{2}} \phi_\delta\left(\frac{x - x_j}{\varepsilon_j^\alpha}, \frac{y}{\varepsilon_j^\alpha}\right) (\varepsilon_j^{\beta-\alpha})^{\frac{2s-n}{2}} w_1\left(\frac{x - x_j}{\varepsilon_j^\alpha \cdot \varepsilon_j^{\beta-\alpha}}, \frac{y}{\varepsilon_j^\alpha \cdot \varepsilon_j^{\beta-\alpha}}\right) \\ &= (\varepsilon_j^\alpha)^{\frac{2s-n}{2}} V_{\varepsilon_j^{\beta-\alpha}}\left(\frac{x - x_j}{\varepsilon_j^\alpha}, \frac{y}{\varepsilon_j^\alpha}\right). \end{aligned}$$

Thus

$$\nabla \widetilde{V}_j(x, y) = \varepsilon_j^{\frac{\alpha(2s-n-2)}{2}} \nabla V_{\varepsilon_j^{\beta-\alpha}}\left(\frac{x - x_j}{\varepsilon_j^\alpha}, \frac{y}{\varepsilon_j^\alpha}\right).$$

Then, by change of variables, we have

$$\begin{aligned} &\int_{\mathbb{R}_+^{n+1}} y^{1-2s} \left(\nabla \widetilde{V}_j\right)^T B(0) \nabla \widetilde{V}_j dx dy \\ (4.1) \quad &= \int_{\mathbb{R}_+^{n+1}} y^{1-2s} \left(\nabla V_{\varepsilon_j^{\beta-\alpha}}(x, y)\right)^T B(0) \nabla V_{\varepsilon_j^{\beta-\alpha}}(x, y) dx dy. \end{aligned}$$

The triangle inequality implies

$$|x| \leq |x - x_j| + |x_j| \leq \delta \varepsilon_j^\alpha + \varepsilon_j.$$

Hence

$$(4.2) \quad \int_{\mathbb{R}_+^{n+1}} y^{1-2s} |x|^\sigma |\nabla \tilde{V}_j|^2 dx dy = O(\varepsilon_j^\sigma).$$

Combining (4.1), (4.2) and applying Lemma 2 we obtain

$$\begin{aligned} \|\tilde{V}_j\|_{H_{0,L}^{s,A}(\mathcal{C}_\Omega)}^2 &\leq \|V_{\varepsilon_j^{\beta-\alpha}}\|_{H_{0,L}^{s,A(0)}(\mathcal{C}_\Omega)}^2 + O(\varepsilon_j^\sigma) \\ &\leq (\det(A(0)))^{\frac{1}{2}} \|\tilde{w}_{\varepsilon_j^{\beta-\alpha}}\|_{H_{0,L}^s(\mathcal{C}_{\tilde{\Omega}})}^2 + O(\varepsilon_j^{(n-2s)(\beta-\alpha)}) + O(\varepsilon_j^\sigma). \end{aligned}$$

Furthermore, by (3.21) and (3.22) we have

$$\int_{\mathbb{R}^n} |\tilde{V}_j(x, 0)|^{\frac{2n}{n-2s}} dx = \int_{\mathbb{R}^n} |V_{\varepsilon_j^{\beta-\alpha}}(x, 0)|^{\frac{2n}{n-2s}} dx \geq K_1 + O(\varepsilon_j^{n(\beta-\alpha)}),$$

and for  $n > 4s$ ,

$$\begin{aligned} \int_{\mathbb{R}^n} |\tilde{V}_j(x, 0)|^2 dx &= \varepsilon_j^{2s\alpha} \int_{\mathbb{R}^n} |V_{\varepsilon_j^{\beta-\alpha}}(x, 0)|^2 dx \\ &= \varepsilon_j^{2s\alpha} \left[ C \varepsilon_j^{2s(\beta-\alpha)} + O(\varepsilon_j^{(n-2s)(\beta-\alpha)}) \right]. \end{aligned}$$

Repeating our argument in Proposition 3, we obtain

$$Q_\lambda^A(\tilde{V}_j) \leq \frac{c_s(\det(A(x_0)))^{\frac{s}{n}} K_s(n)^{-1} - \lambda C \varepsilon_j^{2s\beta} K_1^{-\frac{2(n-2s)}{n+2s}} + O(\varepsilon_j^{(n-2s)(\beta-\alpha)}) + O(\varepsilon_j^\sigma)}{1 + O(\varepsilon^n)}.$$

By our assumption on  $\sigma$ , we can choose  $\beta$  such that

$$(4.3) \quad \frac{\alpha(n-2s)}{n-4s} < \beta < \frac{\sigma}{2s}.$$

It then follows from (4.3) that

$$(4.4) \quad 2s\beta < \min(\sigma, (n-2s)(\beta-\alpha)).$$

(4.3) and (4.4) yield

$$Q_\lambda^A(\tilde{V}_j) < c_s(\det(A(x_0)))^{\frac{s}{n}} K_s(n)^{-1}.$$

## 5. PROOF OF THEOREM 3 - NONEXISTENCE

When  $\lambda$  is nonpositive, our nonexistence result relies on the following Pohozaev identity.

**Lemma 3.** *Assume  $\partial\Omega \in C^1$  and  $a_{ij} \in C^1(\overline{\Omega} \setminus \{x_0\})$ . Let  $A'(x) = (a'_{ij}(x))$  where  $a'_{ij}(x) := \nabla a_{ij}(x) \cdot (x - x_0)$ . Assume further that each  $a'_{ij}$  extends continuously to  $x_0$ . Then for  $u \in C^1(\overline{\Omega})$  and  $w = E_A^s(u)$ , we have*

$$\begin{aligned} &\frac{1}{2} \int_{\partial_L \mathcal{C}_\Omega} y^{1-2s} (\nabla_x w)^T A(x) (\nabla_x w)(x - x_0) \cdot \nu_\Omega d\sigma \\ (5.1) \quad &= \frac{s}{c(s)} \lambda \int_\Omega u^2 dx - \frac{1}{2} \iint_{\mathcal{C}_\Omega} y^{1-2s} (\nabla_x w)^T A'(x) (\nabla_x w) dx dt, \end{aligned}$$

where  $\nu_\Omega$  is the outward normal of  $\partial\Omega$ , and  $d\sigma$  is the area element of  $\partial_L \mathcal{C}_\Omega$ .

*Proof.* Without loss of generality, we assume  $x_0 = 0$ . From approximation arguments in [15], it suffices to prove (5.1) for coefficients  $a_{ij} \in C^1(\overline{\Omega})$  and functions  $u \in C^2(\overline{\Omega})$ . Let  $z := (x, y) \in \Omega \times \mathbb{R}_+$ . Since the matrix  $B(x) = \begin{pmatrix} A(x) & 0 \\ 0 & 1 \end{pmatrix}$  is symmetric, we have

$$\begin{aligned}
 & \operatorname{div} [y^{1-2s} (z \cdot \nabla w) B(x) \nabla w] \\
 &= (z \cdot \nabla w) \operatorname{div} [y^{1-2s} B(x) \nabla w] + [y^{1-2s} B(x) \nabla w]^T \nabla (z \cdot \nabla w) \\
 (5.2) \quad &= (z \cdot \nabla w) \operatorname{div} [y^{1-2s} B(x) \nabla w] + y^{1-2s} (\nabla w)^T B(x) \nabla (z \cdot \nabla w)
 \end{aligned}$$

A direct calculation shows that

$$\begin{aligned}
 & \frac{1}{2} \nabla [(\nabla w)^T B(x) (\nabla w)] \cdot z \\
 (5.3) \quad &= (\nabla w)^T B(x) \nabla (\nabla w \cdot z) - (\nabla w)^T B(x) \nabla w + \frac{1}{2} (\nabla_x w)^T A'(x) \nabla_x w.
 \end{aligned}$$

Combining (5.2) and (5.3), we have

$$\begin{aligned}
 & \operatorname{div} [y^{1-2s} (z \cdot \nabla w) B(x) \nabla w] \\
 &= (z \cdot \nabla w) \operatorname{div} [y^{1-2s} B(x) \nabla w] \\
 & \quad + \frac{y^{1-2s}}{2} \nabla [(\nabla w)^T B(x) \nabla w] \cdot z \\
 (5.4) \quad & \quad + y^{1-2s} (\nabla w)^T B(x) \nabla w - \frac{y^{1-2s}}{2} (\nabla_x w)^T A'(x) \nabla_x w.
 \end{aligned}$$

Integrating both sides of (5.4) over the set  $\mathcal{C}_{R,\varepsilon} := \Omega \times (\varepsilon, R)$  for fixed  $R > \varepsilon > 0$  we obtain

$$\begin{aligned}
 & \int_{\partial \mathcal{C}_{R,\varepsilon}} y^{1-2s} (z \cdot \nabla w) (\nabla w)^T B(x) \nu \, dS \\
 &= \iint_{\mathcal{C}_{R,\varepsilon}} (z \cdot \nabla w) \operatorname{div} [y^{1-2s} B(x) \nabla w] + \frac{y^{1-2s}}{2} \nabla [(\nabla w)^T B(x) (\nabla w)] \cdot z \, dx dy \\
 (5.5) \quad & + \iint_{\mathcal{C}_{R,\varepsilon}} y^{1-2s} (\nabla w)^T B(x) \nabla w - \frac{y^{1-2s}}{2} (\nabla_x w)^T A'(x) \nabla_x w \, dx dy.
 \end{aligned}$$

The first term on the right hand side of (5.5) is zero since  $\operatorname{div} [y^{1-2s} B(x) \nabla w] = 0$  in  $\Omega \times \mathbb{R}_+$ . Integrating by parts for the second term on the right hand side of (5.5)

we derive

$$\begin{aligned}
& \frac{1}{2} \iint_{\mathcal{C}_{R,\varepsilon}} y^{1-2s} \nabla[(\nabla w)^T B(x)(\nabla w)] \cdot z \, dx dy \\
&= -\frac{n+1}{2} \iint_{\mathcal{C}_{R,\varepsilon}} y^{1-2s} (\nabla w)^T B(x)(\nabla w) \, dx dy \\
&\quad -\frac{1-2s}{2} \iint_{\mathcal{C}_{R,\varepsilon}} y^{-2s} \cdot y (\nabla w)^T B(x)(\nabla w) \, dx dy \\
&\quad +\frac{1}{2} \int_{\partial \mathcal{C}_{R,\varepsilon}} y^{1-2s} (\nabla w)^T B(x)(\nabla w)(z \cdot \nu) \, dS \\
&= -\frac{n+2-2s}{2} \iint_{\mathcal{C}_{R,\varepsilon}} y^{1-2s} (\nabla w)^T B(x)(\nabla w) \, dx dy \\
(5.6) \quad & +\frac{1}{2} \int_{\partial \mathcal{C}_{R,\varepsilon}} y^{1-2s} (\nabla w)^T B(x)(\nabla w)(z \cdot \nu) \, dS.
\end{aligned}$$

The boundary integral in (5.6) can be written into three terms:

$$\begin{aligned}
& \int_{\partial \mathcal{C}_{R,\varepsilon}} y^{1-2s} (\nabla w)^T B(x)(\nabla w)(z \cdot \nu_\Omega) \, dS \\
&= \int_{\Omega \times \{y=R\}} y^{2-2s} (\nabla w)^T B(x)(\nabla w) \, dx \\
&\quad - \int_{\Omega \times \{y=\varepsilon\}} y^{2-2s} (\nabla w)^T B(x)(\nabla w) \, dx \\
(5.7) \quad & + \int_{\partial_L \mathcal{C}_{R,\varepsilon}} y^{1-2s} (\nabla w)^T B(x)(\nabla w)(x \cdot \nu_\Omega) \, d\sigma.
\end{aligned}$$

As in (5.7), we write the left hand side of (5.5) into three parts:

$$\begin{aligned}
& \int_{\partial \mathcal{C}_{R,\varepsilon}} y^{1-2s} (z \cdot \nabla w)(\nabla w)^T B(x) \nu \, dS \\
&= \int_{\Omega \times \{y=R\}} y^{1-2s} (z \cdot \nabla w) \frac{\partial w}{\partial y} \, dx - \int_{\Omega \times \{y=\varepsilon\}} y^{1-2s} (z \cdot \nabla w) \frac{\partial w}{\partial y} \, dx \\
&\quad + \int_{\partial_L \mathcal{C}_{R,\varepsilon}} y^{1-2s} (z \cdot \nabla w)(\nabla w)^T B(x) \nu_\Omega \, d\sigma \\
&= \int_{\Omega \times \{y=R\}} y^{1-2s} (z \cdot \nabla w) \frac{\partial w}{\partial y} \, dx - \int_{\Omega \times \{y=\varepsilon\}} y^{1-2s} (z \cdot \nabla w) \frac{\partial w}{\partial y} \, dx \\
(5.8) \quad & + \int_{\partial_L \mathcal{C}_{R,\varepsilon}} y^{1-2s} (\nabla w)^T B(x)(\nabla w)(x \cdot \nu_\Omega) \, d\sigma.
\end{aligned}$$

Here the last equality comes from the fact that  $-\nabla w/|\nabla w| = \nu_\Omega$  on  $\partial_L \mathcal{C}$ . Combining (5.5)-(5.8), we obtain

$$\begin{aligned}
& \int_{\partial_L \mathcal{C}_{R,\varepsilon}} y^{1-2s} (\nabla w)^T B(x) (\nabla w) (x \cdot \nu_\Omega) d\sigma \\
= & \int_{\Omega \times \{y=\varepsilon\}} y^{1-2s} (z \cdot \nabla w) \frac{\partial w}{\partial y} dx - \frac{1}{2} \int_{\Omega \times \{y=\varepsilon\}} y^{2-2s} (\nabla w)^T B(x) (\nabla w) dx \\
& + \frac{2s-n}{2} \iint_{\mathcal{C}_{R,\varepsilon}} y^{1-2s} (\nabla w)^T B(x) (\nabla w) dx dy \\
& - \frac{1}{2} \iint_{\mathcal{C}_{R,\varepsilon}} y^{1-2s} (\nabla_x w)^T A'(x) \nabla_x w dx dy \\
(5.9) \quad & + \frac{1}{2} \int_{\Omega \times \{y=R\}} y^{2-2s} (\nabla w)^T B(x) (\nabla w) dx - \int_{\Omega \times \{y=R\}} y^{1-2s} (z \cdot \nabla w) \frac{\partial w}{\partial y} dx.
\end{aligned}$$

The second term on the right hand side of (5.9) approaches to zero as  $\varepsilon$  does since  $s < 1$ . For the last two terms, there exists  $C > 0$  such that for all  $R \geq 1$ , we have

$$\begin{aligned}
(5.10) \quad & \left| \frac{1}{2} \int_{\Omega \times \{y=R\}} y^{2-2s} (\nabla w)^T B(x) (\nabla w) dx - \int_{\Omega \times \{y=R\}} y^{1-2s} (z \cdot \nabla w) \frac{\partial w}{\partial y} dx \right| \\
& \leq C \int_{\Omega \times \{y=R\}} R^{2-2s} |\nabla w|^2 dx.
\end{aligned}$$

We claim that there exists a sequence  $\{R_i\}$  such that

$$\int_{\Omega \times \{y=R_i\}} R_i^{2-2s} |\nabla w|^2 dx \rightarrow 0 \quad \text{as } R_i \rightarrow \infty.$$

Suppose by contradiction there exist  $a_0 > 0$  and  $R_0 \geq 1$  such that

$$\int_{\Omega \times \{t=R\}} R^{2-2s} |\nabla w|^2 dx \geq a_0 \quad \text{for all } R \geq R_0.$$

Then for any  $R \geq R_0$ ,

$$\int_{\mathbb{R}_+} \int_{\Omega} y^{1-2s} |\nabla w|^2 dx dy \geq \int_{R_0}^R \frac{1}{y} \left[ \int_{\Omega} y^{2-2s} |\nabla w|^2 dx \right] dy \geq a_0 \ln \frac{R}{R_0}.$$

This contradicts  $w \in H_{0,L}^s(\mathcal{C}_\Omega)$ . By (2.5),

$$\begin{aligned}
& \lim_{\varepsilon \rightarrow 0} \int_{\Omega \times \{y=\varepsilon\}} y^{1-2s} (z \cdot \nabla w) \frac{\partial w}{\partial y} dx \\
= & - \frac{1}{c_s} \int_{\Omega \times \{y=0\}} (x \cdot \nabla_x w) (\lambda w + |w|^{\frac{n+2s}{n-2s}}) dx \\
= & - \frac{1}{4c_s} \int_{\Omega \times \{y=0\}} \nabla_x |x|^2 \cdot \nabla_x \left( \lambda w^2 + \frac{n-2s}{n} |w|^{\frac{n+2s}{n-2s}+1} \right) dx \\
= & \frac{n}{2c_s} \int_{\Omega} \lambda u^2 + \frac{n-2s}{n} |u|^{\frac{2n}{n-2s}} dx.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
 & \lim_{R \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} \iint_{\mathcal{C}_{R,\varepsilon}} y^{1-2s} (\nabla w)^T B(x) (\nabla w) \, dx dy \\
 &= \iint_{\mathcal{C}_\Omega} y^{1-2s} (\nabla w)^T B(x) (\nabla w) \, dx dy \\
 (5.11) \quad &= \frac{1}{c_s} \int_{\Omega} \lambda u^2 + |u|^{\frac{2n}{n-2s}} \, dx.
 \end{aligned}$$

Taking  $\varepsilon \rightarrow 0$  and  $R = R_i \rightarrow \infty$  in (5.9) the lemma follows from (5.10)-(5.11).  $\square$

*Proof of Theorem 3.* Suppose by contradiction that problem (1.6) admits a positive solution  $u$  in  $C^1(\overline{\Omega})$ . Then, its extension given by  $w = E_s^A(u) \in C^1(\overline{\mathcal{C}_\Omega})$  is also positive in  $\mathcal{C}_\Omega$  and satisfies  $w = 0$  on  $\partial_L \mathcal{C}_\Omega$ . By the Hopf lemma (see for example [17]),  $\nabla_x w$  is nonzero on  $\partial_L \mathcal{C}_\Omega$ . Note also that the assumption of  $\Omega$  is star-shaped implies  $(x - x_0) \cdot \nu_\Omega > 0$  for all  $x \in \overline{\Omega}$ . Hence, the left-hand side of (5.1) is strictly positive. On the other hand, since  $\lambda \leq 0$  and  $A'(x)$  is positive semi-definite, the right-hand side of (5.1) is non-positive. This contradicts Lemma 3.  $\square$

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